Stochastic Temporal Networks

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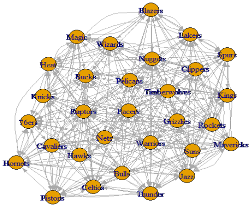
NJIT New Jersey's Science & Technology University

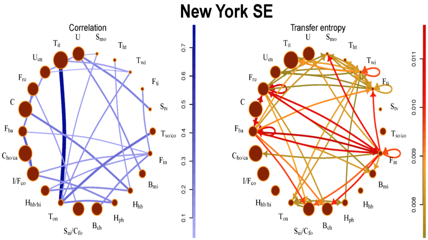
Overview

Static Networks	Temporal Networks
Permanent Interactions	Temporary Interactions

- Transportation
- Internet
- Financial Market



(a)



(b)

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- Generalized Master Equations.
- Laplace Transforms (particularly good for non-Markovian settings)
- Discrete calculus and connections to PDEs.

Discrete-Time Random Walks

X_n walks on graph G with adjacency A_{ij} and transition $P_{ij} = A_{ij}/s_i$ with $s_i = \sum_j A_{ij}$. Define $p_{i;n} = P\left(X_n = i\right)$.

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Applications: centrality measures, modularity (community detections).

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Applications: queueing networks, email communications.

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Transition PDF

$$P_{ij}(t) = \psi_{ij}(t) \prod_{k \neq i} \left(1 - \int_0^t \psi_{ik}(t')\, dt' \right)$$

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which shows that the probability to follow an edge is proportional to its weight λ_{ij}.

In particular, $\frac{dp(t)}{dt} = -Lp(t)$, and we are back to a Poisson random walk on a static graph, where $L_{ij} = \lambda_{ij} - \Lambda_i \delta_{ij}$.

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By a traffic-type equation,

$$\hat{q}(s) = \left(I - \hat{P}(s)\right)^{-1} p(0).$$

Montroll-Weiss Equation

In Laplace space,

$$\hat{p}(s) = \frac{1}{s}\left(I - \hat{D}_P(s)\right)\left(I - \hat{P}(s)\right)^{-1} n(0)\,, \quad \left(\hat{D}_P\right)_{ij}(s) = \hat{P}_i(s)\,\delta_{ij}.$$

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Non-Markovian means nonlocal in time. In real space,

$$\frac{dp}{dt} = \left(P(t) * \mathcal{L}^{-1}\left\{\hat{D}_P^{-1}(s)\right\} - \delta(t)\right) * K(t) * p(t)$$

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Thank you! Questions?

References

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